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SOME BOUNDS ON BALANCED BLOCK DESIGNS.(U)
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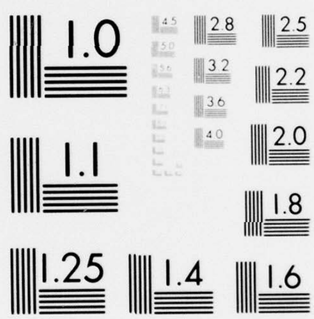
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Some bounds on Balanced Block Designs

by

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March 1978

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Some bounds on balanced block designs

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Abstract

Bounds on the latent root of the C-matrix and the number of blocks for a variance-balanced block design are given. These results include the well known results as special cases.

Some bounds on balanced block designs[1)]

Sanpei Kageyama[2)] and Takumi Tsuji

1. Introduction

Consider v treatments arranged in b blocks with the j-th block being of size k_j $(j = 1,2,\cdots,b)$ in a block design with incidence matrix $N = \|n_{ij}\|$ such that the i-th treatment occurs r_i times $(i = 1, 2,\cdots,v)$ and the i-th treatment occurs in the j-th block n_{ij} times, where n_{ij} can take any of the values, 0, 1, 2,$\cdots$, n-1. Such a design is called an n-ary block design. If n = 2, the design is called a binary block design. Let T_i be the total yield for the i-th treatment and B_j that for the j-th block. On writing $T' = (T_1,\cdots,T_v)$ and $B' = (B_1,\cdots,B_b)$ in matrix notation, the adjusted intrablock normal equations for estimating the vector of treatment effects t can be written under the usual assumptions as

$$Q = C\hat{t} \quad ,$$

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where $\hat{t}$ is the estimate of t,

$$Q = T - N \operatorname{diag}\{k_1^{-1}, k_2^{-1}, \cdots, k_b^{-1}\} B ,$$

$$C = \operatorname{diag}\{r_1, r_2, \cdots, r_v\} - N \operatorname{diag}\{k_1^{-1}, k_2^{-1}, \cdots, k_b^{-1}\} N' ,$$

and diag stands for a diagonal matrix and A' is the transpose of the matrix A, and further let $\operatorname{diag}\{r_1, r_2, \cdots, r_v\} = D_r$ and $\operatorname{diag}\{k_1, k_2, \cdots, k_b\} = D_k$. The matrix C is well known as the C-matrix of a block design.

Since each row (or column) of C adds up to zero, the rank of C is at most v-1, and $(v^{-1/2}, v^{-1/2}, \cdots, v^{-1/2})$ is the latent vector corresponding to the zero root. If the rank of C is v-1, the design is said to be connected (cf. [3]). We shall deal only with connected designs throughout this paper.

A block design is said to be balanced if every elementary contrast of treatments is estimated with the same variance (cf. [11]). In this sense, this design is also called a variance-balanced block (BB) design. Furthermore, it is known (cf. [5], [6], [7], [8], [9], [10], [11]) that an n-ary BB design with parameters v (≥ 2), b (> 0), r_i (> 0), k_j (≥ 2) ($i = 1,2,\cdots,v$; $j = 1,2,\cdots,b$) can be given by an incidence matrix N satisfying

$$(C =)\ D_r - N D_k^{-1} N' = \rho\{I_v - (1/v) G_v\} ,$$

where $\rho = \{\sum_{i=1}^{v} r_i - \sum_{j=1}^{b} (1/k_j) \sum_{i=1}^{v} n_{ij}^2\}/(v-1)$, I_v is the unit matrix of order v, $G_v = E_{v\times v}$ and $E_{\ell\times s}$ is an $\ell \times s$ matrix with positive unit elements everywhere. Note that for a binary BB design, $\rho = (\sum_{i=1}^{v} r_i - b)/(v-1)$.

The literature of block designs contains many articles exclusively related to BB designs. The interested reader can refer, for example, to [5], [6], [7], [8], [9], [10] and [11] for details. Kageyama [7], [8] and [9] has extensively dealt with combinatorial properties and constructions of binary BB designs. In this paper, for an n-ary BB design some bounds on the latent root of the C-matrix and the number of blocks are given. These results include the results well known from various aspects of experimental designs.

Finally, since a design uniquely determines its incidence matrix and vice versa, both a design and its incidence matrix are denoted by the same symbol throughout this paper.

2. Bounds on the latent root and the number of blocks

Let rearrange blocks of a block design N as $N = [N_1 : N_2]$, where N_i's (i = 1,2) consist of some blocks. Then the C-matrix of N can be shown to be

$$C = C_1 + C_2 ,$$

where C_i's (i = 1,2) are the C-matrices of N_i's. Hence, for example, if $C_1 = 0$, then N_1 does not influence discussions on the C-matrix of the design N. Hereinafter we will exclude from our consideration a collection of blocks whose C-matrix is a zero matrix. This collection of blocks can be characterized as follows.

Lemma A. The C-matrix of a collection of some blocks is a zero matrix if and only if each block contains at most only one treatment α ($\geqq 0$) times.

Proof. For a collection of some blocks, let the respective numbers of treatments and blocks be v^* and b^*, and further let the replication numbers of treatments and the sizes of blocks be r_i^* $(i = 1,2,\cdots,v^*)$ and k_j^* $(j = 1,2,\cdots,b^*)$, respectively. We denote the incidence matrix of a collection of b^* blocks by $N^* = \|n_{ij}\|$ $(i = 1,2,\cdots,v^*;\ j = 1,2,\cdots,b^*)$. (Necessity part): $C = \mathrm{diag}\{r_1^*,r_2^*,\cdots,r_{v^*}^*\} - N^*\ \mathrm{diag}\{k_1^{*-1},k_2^{*-1},\cdots,k_{b^*}^{*-1}\}N^{*\prime} = O_{v^*\times v^*}$ is equivalent to $\mathrm{diag}\{r_1^*,r_2^*,\cdots,r_{v^*}^*\} = N^*\ \mathrm{diag}\{k_1^{*-1},\ k_2^{*-1},\cdots,k_{b^*}^{*-1}\}N^{*-1}$ which is expressed as

$$(2.1)\qquad r_i^* = \sum_{j=1}^{b^*} n_{ij}^2/k_j^* \qquad \text{for all } i = 1,2,\cdots,v^*,$$

$$(2.2)\qquad 0 = \sum_{j=1}^{b^*} n_{ij}n_{i'j}/k_j^* \qquad \text{for all } i,\ i'\ (i\neq i') = 1,2,\cdots,v^*,$$

where $O_{s\times t}$ is an $s\times t$ matrix whose elements are all zero. Since $k_j^* > 0$ for all j, (2.2) yields $n_{i1}n_{i'1} = n_{i2}n_{i'2} = \cdots = n_{ib}n_{i'b} = 0$ for all $i,\ i'$ $(i \neq i')$ which imply that each block contains at most only one treatment α times for some α (≥ 0). (Sufficiency part): It obviously follows from the assumption that relations (2.1) and (2.2) holds. Then we have $C = O_{v^*\times v^*}$.

Remark 2.1. From Lemma A, each block of a BB design which will be considered here contains at least two distinct treatments.

The latent roots of the C-matrix play an important role in problems concerning efficiency and analysis for block designs. Especially, as a bound on the latent root, θ, say, for the C-matrix, it is known (cf. [7], [9]) that $0 \leqq \max_{1 \leqq i \leqq v} r_i$ for a general block design. The problem on an improvement of this bound is first considered in this section for the following two cases.

For the convenience of notation, we further let $\max r_i = \max_{1 \leqq i \leqq v} r_i$, $\min r_i = \min_{1 \leqq i \leqq v} r_i$, $\max k_j = \max_{1 \leqq j \leqq b} k_j$ and $\min k_j = \min_{1 \leqq j \leqq b} k_j$.

2.1. For binary BB designs

We first obtain the following bound on the latent root of the C-matrix.

Theorem 2.1.1. For a binary BB design with parameters v, b, r_i, k_j ($i = 1,2,\cdots,v$; $j = 1,2,\cdots,b$) in which $C = \rho\{I_v - (1/v)G_v\}$,

$$\frac{v}{v-1}(\max r_i)\left(1 - \frac{1}{\min k_j}\right) \leqq \rho \leqq \frac{v}{v-1}(\min r_i)\left(1 - \frac{1}{\max k_j}\right)$$

holds.

Proof. Comparing any diagonal element of the C-matrix ($= D_r - ND_k^{-1}N' = \rho\{I_v - (1/v)G_v\}$) yields:

$$\begin{aligned} r_i - \rho\left(1 - \frac{1}{v}\right) &= \frac{n_{i1}}{k_1} + \cdots + \frac{n_{ib}}{k_b} \\ &\geqq \frac{n_{i1} + \cdots + n_{ib}}{\max k_j} \\ &= r_i \,/\, (\max k_j)\ , \quad i = 1,2,\cdots,v, \end{aligned}$$

which implies $\rho \leqq \{v/(v-1)\} r_i \{1 - 1/(\max k_j)\}$ for all $i = 1,2,\cdots,v$.

Hence we get

$$(2.3) \qquad \rho \leqq \frac{v}{v-1}(\min r_i)(1 - \frac{1}{\max k_j}) .$$

On the other hand,

$$r_i - \rho(1 - \frac{1}{v}) = \frac{n_{i1}}{k_1} + \cdots + \frac{n_{ib}}{k_b}$$

$$\leqq \frac{n_{i1} + \cdots + n_{ib}}{\min k_j} = r_i / (\min k_j)$$

which yields $\rho \geqq \{v/(v-1)\}r_i\{1 - 1/(\min k_j)\}$ for all $i = 1,2,\cdots,v$. Hence we get

$$(2.4) \qquad \rho \geqq \frac{v}{v-1}(\max r_i)(1 - \frac{1}{\min k_j}) .$$

Thus, relations (2.3) and (2.4) imply the required result.

Remark 2.2. The upper bound on ρ in Theorem 2.1.1 is attainable if the design is equiblock-sized (in which case, it is obvious that the design is a balanced incomplete block (BIB) design).

For a binary BB design we have the exact value of ρ, i.e., $\rho = (\sum_{i=1}^{n} r_i - b)/(v-1)$. In this sense, the very bound of Theorem 2.1.1 may make no sense practically. However, Theorem 2.1.1 yields a strong restriction on replication numbers r_i $(i = 1,2,\cdots,v)$ as follows.

Corollary 2.1.1. For a binary BB design with parameters v, b, r_i and k_j for $i = 1,2,\cdots,v$ and $j = 1,2,\cdots,b$,

$$\frac{\min r_i}{\max r_i} \geqq \left(\frac{\max k_j}{\max k_j - 1}\right)\left(\frac{\min k_j - 1}{\min k_j}\right) .$$

Remark 2.3. Since $\min k_j \geqq 2$ and $v \geqq \max k_j$, Corollary 2.1.1 further implies that

$$\frac{\min r_i}{\max r_i} \geqq \frac{\max k_j}{2(\max k_j - 1)} \geqq \frac{v}{2(v-1)} .$$

Since $v \geqq \max k_j$ for a binary design, Theorem 2.1.1 yields

Corollary 2.1.2. For a binary BB design with parameters v, b, r_i and k_j $(i = 1,2,\cdots,v;\ j = 1,2,\cdots,b)$ in which $C = \rho\{I_v - (1/v)G_v\}$,

$$\rho \leqq \min r_i .$$

This upper bound is not superior to the upper bound in Theorem 2.1.1. When $v = \max k_j$, both the bounds are the same. However, the bound in Corollary 2.1.2 is very simple and practical. Thus, this bound appears to be worth describing.

2.2. For n-ary BB designs

We here consider bounds on the latent root of the C-matrix for an n-ary BB design. First of all, the bound in Corollary 2.1.2 is not generally valid for an n-ary BB design. For example, we can produce a BB design with parameters $v = 3$, $b = 5$, $r_i = 4$ or 9, $k_j = 4$ or 6, whose incidence matrix is given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 3 & 3 & 0 & 0 & 3 \\ 0 & 0 & 3 & 3 & 3 \end{bmatrix} \quad \text{and} \quad C = (9/2)\{I_3 - (1/3)G_3\} .$$

In this case, $\rho = 9/2 > \min r_i = 4$.

We then describe an upper bound on the latent root of the C-matrix for an n-ary BB design.

Theorem 2.2.1. For an n-ary BB design with parameters v, b, r_i, k_j $(i = 1,2,\cdots,v;\ j = 1,2,\cdots,b)$ in which $C = \rho\{I_v - (1/v)G_v\}$ such that $r_1 \leqq r_2 \leqq \cdots \leqq r_v$,

$$\rho \leqq \min\{\frac{r_1+r_2}{2}, \frac{v}{v-1} r_1 (1 - \frac{1}{\max k_j})\} .$$

Proof. An argument for the former in the proof of Theorem 2.1.1 still holds for an n-ary BB design. We then have $\rho \leqq \{v/(v-1)\}r_1[1 - 1/(\max k_j)]$. Next, from the form of the C-matrix, i.e., $D_r - ND_k^{-1}N'$ $= \rho\{I_v - (1/v)G_v\}$, we get that for any column vector $\underline{x}$,

$$\underline{x}'(D_r - ND_k^{-1}N')\underline{x} = \rho\underline{x}'\{I_v - (1/v)G_v\}\underline{x}$$

which implies that, letting $\underline{x}' = (1/\sqrt{2})(1, -1, 0,\cdots, 0)$,

$$\rho = \frac{1}{2}(r_1 + r_2) - \frac{1}{2}\{\sum_{j=1}^{b}(n_{1j} - n_{2j})^2/k_j\}$$

which yields $\rho \leqq (r_1 + r_2)/2$. Hence the proof is completed.

Remark 2.4. One of upper bounds in Theorem 2.2.1, $\rho \leqq \{v/(v-1)\}$ $(\min r_i)[1 - 1/(\max k_j)]$, attains the bound if $k_1 = k_2 = \cdots = k_b$ and any row (of N) in which $\min r_i$ is attained is binary.

Remark 2.5. From a method similar to Theorem 2.1.1, we can give a lower bound on ρ as follows: $\rho \geqq \{v/(v-1)\} \max_{1\leqq i\leqq v} [r_i\{1-r_i/(\min k_j)\}]$

for an n-ary BB design. However, this bound is meaningful only if there exists an r_i such that $r_i < \min k_j$.

When $\max k_j \leqq v$ in Theorem 2.2.1, it is clear that

$$\frac{r_1 + r_2}{2} \geqq r_1 \geqq \frac{v}{v-1} r_1 \left(1 - \frac{1}{\max k_j}\right) .$$

Then we get

Corollary 2.2.1. For an n-ary BB design with parameters v, b, r_i, k_j in which $C = \rho\{I_v - (1/v)G_v\}$ such that $r_1 \leqq r_2 \leqq \cdots \leqq r_v$,

(i) if $r_1 = r_2$, $\rho \leqq \min\{r_1, \frac{v}{v-1} r_1 (1 - 1/(\max k_j))\}$,

(ii) if $v \geqq \max k_j$, $\rho \leqq \frac{v}{v-1}(\min r_i)(1 - \frac{1}{\max k_j}) \leqq \min r_i$.

Remark 2.6. Each of two conditions, $r_1 = r_2$ and $v \geqq \max k_j$, is a sufficient condition for the validity of the bound $\rho \leqq \min r_i$ $(= r_1)$. We can give other sufficient conditions. For example, from (i) in Corollary 2.2.1, we have only to consider a case in which the cardinality of set $\{i: \min_{1 \leqq i \leqq v} r_i$ is attained$\}$ is one (i.e., $r_1 < r_2 \leqq \cdots \leqq r_v$). In this case, as a sufficient condition for $\rho \leqq r_1$ to be valid, we can present each of the following two conditions: For a BB design $N = \|n_{ij}\|$,

(a) $v\, n_{1j} \geqq k_j$ for all j such that $n_{1j} > 0$;

(b) $v\, (\sum_{j=1}^{b} n_{1j}^2/k_j) \geqq r_1$.

As another upper bound of reflecting certain block structure, we have for an n-ary BB design

(2.5) $$\rho \leqq (v \max_{1\leqq i\neq i'\leqq v} \lambda_{ii'})/(\min k_j) ,$$

where $\lambda_{ii'} = \sum_{j=1}^{b} n_{ij}n_{i'j}$. This can be shown as follows: From Frobenius' theorem (cf. [2], p.66), we have

(2.6) $$\rho \geqq 2 \min_{1\leqq i\leqq v} c_{ii} + (v-2)d ,$$

where c_{ii} is the i-th diagonal element of the C-matrix and d is the numerically largest absolute value of off-diagonal elements of C. Now,

$$|d| = \max_{1\leqq i,i'\leqq v} \{\frac{n_{i1}n_{i'1}}{k_1} + \cdots + \frac{n_{ib}n_{i'b}}{k_b}\}$$

(2.7) $$\leqq \max_{i,i'} \{\frac{n_{i1}n_{i'1} + \cdots + n_{ib}n_{i'b}}{\min k_j}\}$$

$$= (\max_{i,i'} \lambda_{ii'})/(\min k_j) ,$$

where $\lambda_{ii'} = \sum_{j=1}^{b} n_{ij}n_{i'j}$. Since $c_{ii} = \rho(1 - 1/v)$ and $v \geqq 2$, we get (2.5) from (2.6) and (2.7). However, bound (2.5) may be not relatively good as an upper bound.

Furthermore, we can present mathematically an upper bound on ρ which gives a partial improvement of Theorem 2.2.1. The following result also plays an important role on an argument (of Section 2.3) providing sufficient conditions for the validity of Fisher's inequality.

Theorem 2.2.2. For an n-ary BB design with parameters v, b, r_i, k_j ($i = 1,2,\cdots,v$; $j = 1,2,\cdots,b$) in which $C = \rho\{I_v - (1/v)G_v\}$,

$$\rho \leqq \rho_0 ,$$

where ρ_0 is the least positive root of the following polynomial of degree v-1

$$(2.8) \qquad f(\rho) = |D_r - \rho I_v + (\rho/v)G_v|$$

$$= \frac{(-1)^{v-1}}{v}\left(\sum_{i=1}^{v} r_i\right)\rho^{v-1} + (-1)^{v-2}\frac{2}{v}\left(\sum_{i<j} r_i r_j\right)\rho^{v-2}$$

$$+ (-1)^{v-3}\frac{3}{v}\left(\sum_{i<j<k} r_i r_j r_k\right)\rho^{v-3} + \cdots + \frac{v}{v} r_1 r_2 \cdots r_v .$$

The proof of this theorem needs some preliminary results. The following two lemmas are available in various books on linear algebra.

Lemma 2.2.1 (cf. [1], p.75). For a real symmetric matrix $A = \|a_{ij}\|$ of order v, A is positive definite if and only if

$$|A^{(s)}| > 0 \quad \text{for} \quad s = 1,2,\cdots,v ,$$

where

$$A^{(s)} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{ss} \end{bmatrix} .$$

Lemma 2.2.2 (cf. [1], p.117). When A is a real symmetric matrix and B is any principal submatrix of A, the maximal latent root of A is greater than or equal to the maximal latent root of B.

Proposition 2.2.1. There exists the least positive root (= ρ_0, say) of $f(\rho)$ in (2.8). (i) If $\rho < \rho_0$, then $D_r - \rho I_v + (\rho/v)G_v$ is positive definite. (ii) If $\rho = \rho_0$, then $D_r - \rho I_v + (\rho/v)G_v$ is positive semidefinite and singular. (iii) If $\rho > \rho_0$, then $D_r - \rho I_v + (\rho/v)G_v$ is not positive semidefinite.

Proof. Let $A^{(s)} = D_r^{(s)} - \rho I_s + (\rho/v)G_s$ for $s = 1,2,\cdots,v$ which is a principal submatrix of $D_r - \rho I_v + (\rho/v)G_v$ $(= ND_k^{-1}N')$, where $D_r^{(s)} = \mathrm{diag}\{r_1$ $r_2,\cdots,r_s\}$. Further, let $f^{(s)}(\rho) = |A^{(s)}|$ for $s = 1,2,\cdots,v$. In particular, $f^{(v)}(\rho) = f(\rho)$ in (2.8). Now, consider roots of $f^{(s)}(\rho)$ $= |D_r^{(s)} - \rho\{I_s - (1/v)G_s\}| = 0$ which also yields $\rho \neq 0$. Then the nonzero roots of $f^{(s)}(\rho)$ can be shown to be equivalent to the nonzero roots of the following equation:

$$(2.9) \qquad |(1/\rho)I_s - D_r^{(s)^{-1/2}}\{I_s - (1/v)G_s\}D_r^{(s)^{-1/2}}| = 0$$

for $s = 1,2,\cdots,v$. Furthermore, $D_r^{(s)^{-1/2}}\{I_s - (1/v)G_s\}D_r^{(s)^{-1/2}}$ is positive semidefinite. Hence its latent root, $1/\rho$, is real and non-negative, i.e., ρ is a positive real number. Hence $f^{(s)}(\rho)$ has only positive roots. Let $\rho_0^{(s)}$ be the least positive root of $f^{(s)}(\rho)$. Especially, $\rho_0^{(v)}$ $(= \rho_0$, say) is the least positive root of $f(\rho)$. In this case, we can show that

$$(2.10) \qquad \rho_0^{(1)} \geqq \rho_0^{(2)} \geqq \cdots \geqq \rho_0^{(v-1)} \geqq \rho_0^{(v)} = \rho_0 .$$

This can be given as follows. In (2.9), $D_r^{(s-1)^{-1/2}}\{I_{s-1} - (1/v)G_{s-1}\}$ $\cdot D_r^{(s-1)^{-1/2}}$ is obviously a principal submatrix of $D_r^{(s)^{-1/2}}\{I_s - (1/v)G_s\}$ $\cdot D_r^{(s)^{-1/2}}$. This fact together with Lemma 2.2.2 implies that the maximal latent root of $D_r^{(s)^{-1/2}}\{I_s - (1/v)G_s\}D_r^{(s)^{-1/2}}$ is greater than or equal to the maximal latent root of $D_r^{(s-1)^{-1/2}}\{I_{s-1} - (1/v)G_{s-1}\}D_r^{(s-1)^{-1/2}}$.

This statement yields (2.10).

(i) Since $f^{(s)}(0) > 0$ and $f^{(s)}(\rho)$ is a polynomial, if $\rho < \rho_0$, then, from the meaning of ρ_0, $f^{(s)}(\rho) > 0$ holds for $s = 1,2,\cdots,v$. Hence, from Lemma 2.2.1, $D_r - \rho I_v + (\rho/v)G_v$ is positive definite.

(ii) If $\rho = \rho_0$, then $f(\rho_0) = 0$, i.e., $D_r - \rho_0 I_v + (\rho_0/v)G_v$ is singular. For any nonzero column vector $\underline{x}$, let $\underline{x}'\{D_r - \rho I_v + (\rho/v)G_v\}\underline{x} = g(\rho : \underline{x})$. Then $g(\rho : \underline{x})$ is continuous linear function on ρ and, from (i), $g(\rho : \underline{x}) > 0$ for $\rho < \rho_0$. Thus, $g(\rho_0 : \underline{x}) \geq 0$ for any nonzero vector $\underline{x}$. Therefore, $D_r - \rho_0 I_v + (\rho_0/v)G_v$ is positive semidefinite.

(iii) Since $D_r - \rho_0 I_v + (\rho_0/v)G_v$ is singular, there exists a nonzero column vector $\underline{x}$ such that $\{D_r - \rho_0 I_v + (\rho_0/v)G_v\}\underline{x} = \underline{0}$. In this case, we get

$$\begin{aligned}
&\underline{x}'\{D_r - \rho I_v + (\rho/v)G_v\}\underline{x} \\
&\qquad = \underline{x}'\{D_r - \rho_0 I_v + (\rho_0/v)G_v\}\underline{x} + (\rho_0 - \rho)\underline{x}'\{I_v - (1/v)G_v\}\underline{x} \\
&\qquad = (\rho_0 - \rho)\underline{x}'\{I_v - (1/v)G_v\}\underline{x} \\
&\qquad = (\rho_0 - \rho)\underline{x}'\{(1/\rho_0)D_r\}\underline{x} < 0 ,
\end{aligned}$$

since $\rho > \rho_0$. Therefore, $D_r - \rho I_v + (\rho/v)G_v$ is not positive semidefinite.

Proof of Theorem 2.2.2. From the C-matrix of the design, $D_r - \rho I_v + (\rho/v)G_v = ND_k^{-1}N'$ is positive semidefinite. Hence Proposition 2.2.1 completes the proof.

We also give examples showing the goodness of respective upper bounds in Theorems 2.2.1 and 2.2.2.

Example 2.1. Consider a BB design with parameters $v = 5$, $b = 8$, $r_i = 4$ or 8, $k_j = 3$, whose incidence matrix is given by

$$\begin{bmatrix} 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \frac{10}{3}(I_5 - \frac{1}{5}G_5) .$$

In this case, $\rho \leqq (r_1 + r_2)/2$, $\rho \leqq [v/(v-1)]r_1[1 - 1/(\max k_j)]$ and $\rho \leqq \rho_0$ imply $\rho \leqq 4$, $\rho \leqq 10/3$ and $\rho \leqq 4$, respectively. Thus, $\min r_i = \rho_0$.

Example 2.2. Consider an example described before Theorem 2.2.1. For this case, $\rho \leqq (r_1 + r_2)/2$, $\rho \leqq [v/(v-1)]r_1[1 - 1/(\max k_j)]$ and $\rho \leqq \rho_0$ imply $\rho \leqq 13/2$, $\rho \leqq 5$ and $\rho \leqq 54/11$, respectively.

For a notation of Proposition 2.2.1, we have $\rho_0^{(1)} = \{v/(v-1)\}r_1$ and $\rho_0^{(2)} = \{(v-1)(r_1 + r_2) - \sqrt{(v-1)^2(r_1 + r_2)^2 - 4r_1r_2v(v-2)}\}/2(v-2)$ $= \{(v-1)(r_1 + r_2) - \sqrt{(v-1)^2(r_1 - r_2)^2 + 4r_1r_2}\}/2(v-2)$. Hence

$$\rho_0 \leqq \frac{(v-1)(r_1 + r_2) - \sqrt{(v-1)^2(r_1 - r_2)^2 + 4r_1r_2}}{2(v-2)} .$$

Furthermore, it can be easily shown that

$$\frac{(v-1)(r_1 + r_2) - \sqrt{(v-1)^2(r_1 - r_2)^2 + 4r_1r_2}}{2(v-2)} \leqq \frac{r_1 + r_2}{2}$$

which implies that $\rho_0 \leqq (r_1 + r_2)/2$. Thus, Theorem 2.2.2 gives a partial improvement of Theorem 2.2.1.

Corollary 2.2.2. For an n-ary BB design with parameters v, b, r_i, k_j ($i = 1,2,\cdots,v$; $j = 1,2,\cdots,b$) in which $C = \rho\{I_v - (1/v)G_v\}$ such that $r_1 \leqq r_2 \leqq \cdots \leqq r_v$,

$$\rho \leqq \frac{(v-1)(r_1+r_2) - \sqrt{(v-1)^2(r_1-r_2)^2+4r_1r_2}}{2(v-2)}$$

holds.

Note that if $r_1 = r_2$, Corollary 2.2.2 yields $\rho \leqq \min r_i$ $(= r_1)$.

We now compare the value, ρ_0, with an interesting value, $\min r_i$. The result of this comparison will be used later.

Lemma 2.2.3. For an n-ary BB design with parameters v, b, r_i, k_j $(i = 1,2,\cdots,v;\ j = 1,2,\cdots,b)$,

$$\min r_i \leqq \rho_0 ,$$

where ρ_0 is the least positive root of $f(\rho)$ in (2.8).

Proof. Assume, without loss of generality, that $r_1 \leqq r_2 \leqq \cdots \leqq r_v$. Now, consider the following matrix for any ε such that $0 < \varepsilon < r_1$.

$$\begin{aligned} D_r &- (r_1 - \varepsilon)I_v + \{(r_1-\varepsilon)/v\}G_v \\ &= \mathrm{diag}\{\varepsilon, r_2-r_1+\varepsilon, \cdots, r_v-r_1+\varepsilon\} + \{(r_1-\varepsilon)/v\}G_v , \end{aligned}$$

in which case $\mathrm{diag}\{\varepsilon, r_2-r_1+\varepsilon, \cdots, r_v-r_1+\varepsilon\}$ is positive definite and $\{(r_1-\varepsilon)/v\}G_v$ is positive semidefinite. Thus, $D_r - (r_1-\varepsilon)I_v+\{(r_1-\varepsilon)/v\}G_v$ is positive definite. Hence, from Proposition 2.2.1, we obtain $\rho_0 > r_1-\varepsilon$. Since ε is arbitrary $(0 < \varepsilon < r_1)$, $\rho_0 \geqq r_1 = \min r_i$.

2.3. Conditions for Fisher's inequality

We here consider bounds on the number of blocks in a BB design.

Theorem 2.3.1. In an n-ary BB design with parameters v, b, r_i, k_j $(i = 1,2,\cdots,v;\ j = 1,2,\cdots,b)$ in which $C = \rho\{I_v - (1/v)G_v\}$, if $\rho < \rho_0$, then $b \geqq v$ holds, where ρ_0 is the least positive root of $f(\rho)$ in (2.8).

Proof. From (i) of Proposition 2.2.1, if $\rho < \rho_0$, then $D_r - \rho I_v + (\rho/v)G_v$ $(= ND_k^{-1}N')$ is positive definite. Hence $v = \text{rank } ND_k^{-1}N' = \text{rank } N \leqq b$.

In Lemma 2.2.3, we have $\min r_i \leqq \rho_0$. This fact together with Theorem 2.3.1 implies

Corollary 2.3.1. For an n-ary BB design with parameters v, b, r_i and k_j $(i = 1,2,\cdots,v;\ j = 1,2,\cdots,b)$ in which $C = \rho\{I_v - (1/v)G_v\}$, if $\rho < \min r_i$, then $b \geqq v$ holds.

Note that if there exists only one i such that $\min r_i$ is attained, then the sufficient condition for Fisher's inequality to be valid can be improved to $\rho \leqq \min r_i$.

From (ii) of Corollary 2.2.1, if $v > \max k_j$, then $\rho < \min r_i$ holds and hence we have

Corollary 2.3.2. For an n-ary BB design with parameters v, b, r_i and k_j $(i = 1,2,\cdots,v;\ j = 1,2,\cdots,b)$ in which $C = \rho\{I_v - (1/v)G_v\}$, if $v > \max k_j$, then $b \geqq v$ holds.

As a characterization of a special case in which the bound of Corollary 2.1.2 is attainable, we have

Theorem 2.3.2. A binary BB design N with parameters v, b, r_i and k_j ($\geqq 2$) ($i = 1,2,\cdots,v$; $j = 1,2,\cdots,b$) and $C = \rho\{I_v - (1/v)G_v\}$ satisfies $\rho = \min r_i$ if and only if the design is a complete block design (i.e., $N = E_{v\times b}$).

Proof. It is obvious that the sufficiency part is valid. We then consider only the necessity part. For the C-matrix of a binary BB design N such that $r_1 \leqq r_2 \leqq \cdots \leqq r_v$, when $\rho = r_1$ (= $\min r_i$), we have

$$ND_k^{-1}N' = D_r - \rho I_v + (\rho/v)G_v$$

$$= \begin{bmatrix} \frac{\rho}{v} & \frac{\rho}{v} & \cdots & \frac{\rho}{v} \\ \frac{\rho}{v} & r_2-\rho+\frac{\rho}{v} & & \\ \vdots & & \ddots & \frac{\rho}{v} \\ & \frac{\rho}{v} & & \\ \frac{\rho}{v} & & & r_v-\rho+\frac{\rho}{v} \end{bmatrix} = \|m_{ij}\| \text{ , say,}$$

for $i, j = 1,2,\cdots,v$. Since $m_{11} = m_{12} = \cdots = m_{1v}$ (= ρ/v), we get

$$(2.11) \qquad 0 = \frac{n_{11}(1-n_{i1})}{k_1} + \frac{n_{12}(1-n_{i2})}{k_2} + \cdots + \frac{n_{1b}(1-n_{ib})}{k_b} \quad \text{for } i \geqq 2.$$

Since $b \geqq r_1$, we can further assume, without loss of generality, that

$$(2.12) \qquad n_{11} = n_{12} = \cdots = n_{1r_1} = 1 \text{ and } n_{1r_1+1} = \cdots = n_{1b} = 0.$$

Relations (2.11) and (2.12) imply

$$n_{i1} = n_{i2} = \cdots = n_{ir_1} = 1 \qquad \text{for all } i = 2,3,\cdots,v.$$

Thus,

$$N = \left[\begin{array}{c|c} & 0\ 0 \cdots 0 \\ E_{v\times r_1} & \cdot\ * \\ & \end{array}\right] \qquad (= [N_1 : N_2],\ \text{say}),$$

the C-matrix of which is given by $C = C_1 + C_2$, where C_i's ($i = 1,2$) are the C-matrices of N_i's. Here, $C = C_1 = r_1\{I_v - (1/v)G_v\}$ and then $C_2 = 0$. Hence, from Lemma A, N_2 cannot happen for this design N. Thus, we must have $r_1 = b$ and then $N = E_{v\times b}$.

Remark 2.7. From a method of proving Theorem 2.3.2, we can also deduce that a binary BB design with parameters v, b, r_i, k_j ($i = 1,2, \cdots,v$; $j = 1,2,\cdots,b$) and with $C = \rho\{I_v - (1/v)G_v\}$ satisfies $\rho = r_i$ for some i if and only if the design is a complete block design. Furthermore, note (cf. [4]) that in an n-ary BB design N with parameters v, b, r and k_j ($j = 1,2,\cdots,b$) and with $C = \rho\{I_v - (1/v)G_v\}$, $\rho = r$ holds if and only if each row of N is equal.

As seen from Remark 2.7, a BB design is usually considered in the following case, aside from trivialities: (i) $\rho < r_i$ for all i in a binary BB design. (ii) $\rho < r$ in an n-ary equireplicated BB design.

From Corollaries 2.1.2 and 2.3.1, and Theorem 2.3.2, we obtain an useful result:

Corollary 2.3.3. For a binary BB design with parameters v, b, r_i, k_j ($\geqq 2$) ($i = 1,2,\cdots,v$; $j = 1,2,\cdots,b$) which is not of type $E_{v\times b}$, $b \geqq v$ holds.

Incidentally, as more general bounds on the number of blocks which are different from Fisher's inequality, we can get

Theorem 2.3.3. For a binary BB design with parameters v, b, r_i, k_j and $n = \sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j$,

$$n - \left(1 - \frac{1}{\max k_j}\right)(\min r_i)v \leqq b \leqq n - \left(1 - \frac{1}{\min k_j}\right)(\max r_i)v .$$

Furthermore, if the design is equireplicated (i.e., $r_1 = \cdots = r_v = r$, say), then

$$\left(\frac{r}{\max k_j}\right)v \leqq b \leqq \left(\frac{r}{\min k_j}\right)v .$$

Proof. From a comparison of the i-th diagonal element of the C-matrix ($= D_r - ND_k^{-1}N' = \rho\{I_v - (1/v)G_v\}$) of a binary BB design with parameters v, b, r_i, k_j and $\rho = (n-b)/(v-1)$,

$$r_i - \left(\frac{n_{i1}^2}{k_1} + \cdots + \frac{n_{ib}^2}{k_b}\right) = \frac{n-b}{v} \quad \text{for all } i \geqq 1$$

i.e.,

$$\text{(2.13)} \qquad r_i = \frac{n-b}{v} + \frac{n_{i1}^2}{k_1} + \cdots + \frac{n_{ib}^2}{k_b} , \; i = 1,2,\cdots,v .$$

Relation (2.13) can be evaluated in two ways. First,

$$r_i \geqq \frac{n-b}{v} + \frac{n_{i1} + \cdots + n_{ib}}{\max k_j}$$

$$= \frac{n-b}{v} + \frac{r_i}{\max k_j} ,$$

which yields $(\min r_i)\{1 - 1/(\max k_j)\} \geqq (n-b)/v$. Hence we have

$$b \geqq n - \{1 - 1/(\max k_j)\}(\min r_i)v \ .$$

When $r_1 = r_2 = \cdots = r_v = r$, say, we also have

$$b \geqq \{r/(\max k_j)\}v \ ,$$

since $n = vr$. Next,

$$r_i \leqq \frac{n-b}{v} + \frac{n_{i1} + \cdots + n_{ib}}{\min k_j}$$

$$= \frac{n-b}{v} + \frac{r_i}{\min k_j} \ .$$

Similarly, we can get

$$b \leqq n - \{1 - 1/(\min k_j)\}(\max r_i)v \ .$$

When $r_1 = r_2 = \cdots = r_v = r$, say, we also have $b \leqq \{r/(\min k_j)\}v$.

The last bound of Theorem 2.3.3 is obvious, but combinatorially interesting. Note that Theorem 2.3.3 still holds for a binary partially balanced block (PBB) design (see [7] for the definition of a PBB design).

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